2.7 Derivative and Rates of Change

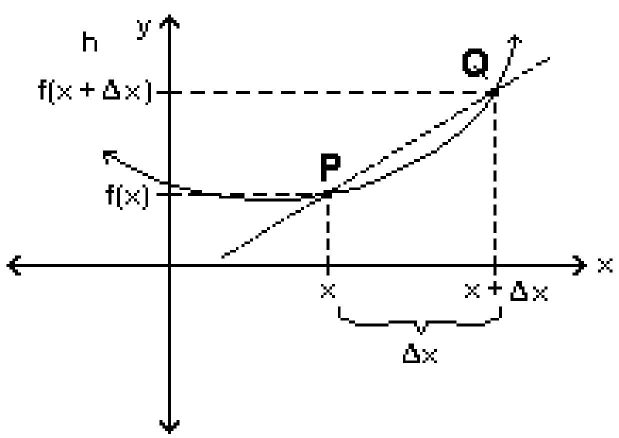
In this section we will visit the ideas of a tangent line, instantaneous velocity and introduce the derivative.

In section 2.1, the tangent line of a curve with equation ***y = f(x)*** at point (a, f(a)) was found by considering nearby points, such that ***x≠a***, and compute the slope of the secant line. Then we approach point (a, f(a)) by letting ***x*** approach ***a***. If we get infinitely close to ***a***, then we can find the slope of the curve ***y = f(x)*** at ***a***. In other words we would get the following:

Definition: The tangent line to the curve ***y = f(x)*** at the point P(a, f(a)) is the line through P with slope

$$\mathbf{m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}} \text{ provided that the limit exits.}$$

Similar to the definition above, we also have the following: $\mathbf{m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}}$ where $\Delta \boldsymbol{x}$ is the distance that gets smaller (closer to x). Graphically, this is what we have.



Example: Using $\mathbf{m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}}$ find the equation of the tangent line to $\boldsymbol{y} = \sqrt{\boldsymbol{x}}$ at the point (4, 2). (Since Δx is cumbersome to use we can use $\boldsymbol{h}$ in place of Δx.) This would make the formula in the definition above look like: $\mathbf{m} = \mathbf{\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$.

Let ***f(x)*** $= \sqrt{x}$ and ***x = 4***.

$$m = \lim_{h \to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h} = \lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$$= \lim_{h \to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} = \lim_{h \to 0} \frac{h}{h(\sqrt{4+h}+2)} = \lim_{h \to 0} \frac{1}{(\sqrt{4+h}+2)}$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{\mathbf{1}}{\mathbf{4}}$$ therefore the slope of the tangent line is $\frac{\mathbf{1}}{\mathbf{4}}$.

Now let's use the slope of $\frac{1}{4}$ and the given point (4, 2) and write an equation of the tangent line.

I use the point-slope method: $y - y_1 = m(x - x_1)$

$$y - 2 = \frac{1}{4}(x-4)$$

$$y = \frac{1}{4}x - 1 + 2 \Rightarrow \quad y = \frac{1}{4}x + 1$$

Now thinking about velocity (or instantaneous velocity) ***V(x)*** at time ***t = x*** we use a similar idea as that of secant lines to tangent lines and we are able to see how the instantaneous velocity is derived from average velocities. Since the derivation of instantaneous velocity is similar to that of a tangent line starting from a secant line, we will introduce the following without more explanation. ***V(x)*** is the instantaneous velocity if:

$$\boldsymbol{V(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

Example: If a ball is thrown into the air with velocity of 40 ft/sec, its height (in feet) is given by $y = 40t - 16t^2$. Find the instantaneous velocity when t = 2.

$V(2) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$ where $t = 2$ and $y = f(t)$

$$\begin{aligned} f(t) = f(2) &= 40(2) - 16(2)^2 \\ &= 80 - 64 \\ &= 16 \end{aligned} \qquad \begin{aligned} f(t+h) = f(2+h) &= 40(2+h) - 16(2+h)^2 \\ &= 80 + 40h - 16(4 + 4h + h^2) \\ &= 80 + 40h - 64 - 64h - 16h^2 \end{aligned}$$

Thus: $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{80+40h-64-64h-16h^2-16}{h} = \lim_{h\to 0}\frac{-24h-16h^2}{h} = \lim_{h\to 0}\frac{h(-24-2h)}{h}$

$= \lim_{h\to 0} -24 - 2h = \mathbf{-24}$

Therefore, the instantaneous velocity *V(2)= -24 feet per second.*

We have seen that the same type of limit arises in the finding of the slope of the tangent line. Limits of the form $\boldsymbol{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$ arise whenever we calculate a rate of change in any of the sciences. We will call this the **derivative.**

Definition: The **derivative** of a function ***f*** at a number ***a***, denotes by $\boldsymbol{f'(a)}$ is:

$$\boldsymbol{f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}} \quad \text{provided the limit exists.}$$

Example: Find the derivative of the function $f(x) = x^2 + 4x - 3$ at the number ***a***.

From the definition we have
$$\begin{aligned} \boldsymbol{f'(a)} &\boldsymbol{= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}} \\ &\boldsymbol{= \lim_{h\to 0}\frac{[(a+h)^2+4(a+h)-3]-[a^2+4a-3]}{h}} \\ &\boldsymbol{= \lim_{h\to 0}\frac{a^2+2ah+h^2+4a+4h-3-a^2-4a+3}{h}} \\ &\boldsymbol{= \lim_{h\to 0}\frac{2ah+h^2+4h}{h}} \\ &\boldsymbol{= \lim_{h\to 0}\frac{h(2a+h+4)}{h}} \\ &\boldsymbol{= \lim_{h\to 0} 2a + h + 4} \end{aligned}$$

$$\boldsymbol{f'(a) = 2a + 4}$$

Using the definition of the derivative we also get the following: The Tangent line to ***y = f(x)*** at ***(a, f(a))*** is the line through ***(a, f(a))*** whose slope is equal to ***f '(a)***, the derivative of ***f*** at ***a.***

Example: Find the equation of the tangent line to the parabola $\boldsymbol{y = x^2 + 4a - 3}$ at the point (1, 2).

From the last example we saw that the derivative of $\boldsymbol{y = x^2 + 4a - 3}$ at a is $\boldsymbol{f'(a) = 2a + 4.}$ Therefore, the slope of the tangent line at (1, 2) is $f'(1) = 2(1) + 4 = \mathbf{6}.$ (Note that when the point changed, the slope of the tangent line changed.) Now we find the equation of the tangent line as before.

$$y - 2 = 6(x - 1)$$
$$y - 2 = 6x - 6$$
$$\boldsymbol{y = 6x - 4}$$

Rates of Change.

Suppose **y** is a quantity that depends on another quantity **x.** Thus **y** is a function of **x** and we write **y = f(x).** If we change from x_1 *to* x_2, then the change in **x** (Δx) is

$$\boldsymbol{\Delta x = x_2 - x_1}$$

and the corresponding change in **y** (Δy) is

$$\boldsymbol{\Delta y = f(x_2) - f(x_1)}$$

The Difference Quotient $\boldsymbol{\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}}$ is called the **average rate of change** of **y** with respect to **x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line.

The **limit** of these **average rates of change** is called the **instantaneous rate of change** of ***y*** with respect to ***x*** at **x = x**$_1$.

$$\boldsymbol{Instantaneous\ Rate\ of\ Change = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

The derivative ***f '(a)*** is the instantaneous rate of change of ***y = f(x)*** with respect to **x** when **x = a.**

For applications, if ***s = f(t)*** is the position function of a particle that moves along a straight line, then ***f '(a)*** is the rate of change of the displacement, ***s***, with respect to the time ***t***. In other words, ***f '(a)*** is the velocity of the particle at time ***t = a***. The speed of the particle is the absolute value of the velocity, **|*f '(a)*|**.

Example: The cost of producing **x** units of a certain commodity is $\boldsymbol{C(x) = 5000 + 10x + 0.05x^2}$.
(a) Find the average rate of change of **C** when **x** changes from **x = 100** to **x = 105.**
(b) Find the instantaneous rate of change **C** when x = 100. (Marginal Cost)

(a) Average rate of change $= \frac{C(105)-C(100)}{105-100} = \frac{6601.3-6500}{5} = \frac{101.3}{5} = \$20.25/unit$

(b) Instantaneous Rate of Change: $C'(100) = \lim_{h\to 0} \frac{C(100+h)-C(100)}{h}$

$$= \lim_{h \to 0} \frac{[5000+10(100+h)+0.05\left(100+h\right)^2]-[5000+1000+500]}{h}$$

$$= \lim_{h \to 0} \frac{5000+1000+10h+0.05(10000+200h+h_2)-5000-1000-500}{h}$$

$$= \lim_{h \to 0} \frac{6500+20h+0.05h^2-6500}{h}$$

$$= \lim_{h \to 0} \frac{20h+0.05h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(20+0.05h)}{h}$$

$$= \lim_{h \to 0} 20 + 0.05h$$

$= \mathbf{\$20/unit}$